Influence of susceptibility - probability distributions

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If $Z \sim N\left(\mu, \sigma^2\right)$ then $E\left(e^z\right) = \int_{-\infty}^{+\infty} e^z \varphi\left(z\right) dz$ with $\varphi\left(z\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

$$
\begin{aligned}
E\left(e^z\right) &= \int_{-\infty}^{+\infty} e^z \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^z e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \quad (1)
\end{aligned}
$$

$$
\begin{aligned}
z - \frac{(z-\mu)^2}{2\sigma^2} &= \frac{1}{2}\left(\frac{-(z-\mu)^2}{\sigma^2} + 2z\right) \\
&= \frac{1}{2}\left(\frac{-(z-\mu)^2}{\sigma^2} + 2(z-\mu) - \sigma^2\right) + \mu + \frac{\sigma^2}{2} \\
&= \frac{-1}{2}\left(\frac{z-\mu}{\sigma} - \sigma\right)^2 + \mu + \frac{\sigma^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
(1) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma} - \sigma\right)^2} e^{\mu + \frac{\sigma^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}\sigma} e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma} - \sigma\right)^2} dz \quad (2)
\end{aligned}
$$

Put $w = \frac{z-\mu}{\sigma} - \sigma \Rightarrow dw = \frac{1}{\sigma} dz \Rightarrow dz = \sigma dw$.

$$
\begin{aligned}
(2) &= \frac{1}{\sqrt{2\pi}\sigma} e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}} \sigma dw \\
&= \frac{1}{\sqrt{2\pi}} e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}} dw \\
&= \frac{1}{\sqrt{2\pi}} e^{\mu + \frac{\sigma^2}{2}} \sqrt{2\pi} \\
&= e^{\mu + \frac{\sigma^2}{2}}
\end{aligned}
$$

$$Var\left(e^z\right) = E\left[e^{2z}\right] - \left(E\left[e^z\right]\right)^2$$

$$
\begin{aligned}
E\left[e^{2z}\right] &= \int_{-\infty}^{+\infty} e^{2z} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{2z - \frac{(z-\mu)^2}{2\sigma^2}} dz
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
2z - \frac{(z-\mu)^2}{2\sigma^2} &= \frac{1}{2}\left(\frac{(z-\mu)^2}{\sigma^2} + 4z\right) \\
&= \frac{1}{2}\left(-\frac{(z-\mu)^2}{\sigma^2} + 4(z-\mu) - 4\sigma^2\right) + 2\mu + 2\sigma^2 \\
&= \frac{-1}{2}\left(\frac{z-\mu}{\sigma} - 2\sigma\right)^2 + 2\mu + 2\sigma^2
\end{aligned}
$$

$$
(3) = \frac{1}{\sqrt{2\pi}\sigma} e^{2\mu + 2\sigma^2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\mu}{\sigma} - 2\sigma\right)^2} dz
\tag{4}
$$

Put $w = \frac{z-\mu}{\sigma} - 2\sigma \Rightarrow dw = \frac{dz}{\sigma} \Rightarrow dz = \sigma dw$.

$$
\begin{aligned}
(4) &= \frac{1}{\sqrt{2\pi}\sigma} e^{2\left(\mu + \sigma^2\right)} \int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}} \sigma dw \\
&= \frac{1}{\sqrt{2\pi}} e^{2\left(\mu + \sigma^2\right)} \int_{-\infty}^{+\infty} e^{\frac{-w^2}{2}} dw \\
&= \frac{1}{\sqrt{2\pi}} e^{2\left(\mu + \sigma^2\right)} \sqrt{2\pi} \\
&= e^{2\left(\mu + \sigma^2\right)}
\end{aligned}
$$

$$
\begin{aligned}
Var\left(e^z\right) &= e^{2\left(\mu + \sigma^2\right)} - e^{2\mu + \sigma^2} \\
&= \left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2}
\end{aligned}
$$

$E\left(e^z\right) = 1 \Leftrightarrow \mu = -\frac{\sigma^2}{2}$

$b_0 \sim N\left(\mu_0, \sigma_0^2\right)$ and $b_1 \sim N\left(\mu_1, \sigma_0^1\right)$

The following conditions have to be fulfilled:

$E\left(e^{b_0 + b_1}\right) = e^{\mu_0 + \mu_1 + \frac{\left(\sigma_0^2 + \sigma_1^2\right)}{2}} = 1$

$E\left(e^{b_0 + b_2}\right) = e^{\mu_0 + \mu_2 + \frac{\left(\sigma_0^2 + \sigma_2^2\right)}{2}} = 1$

This is a.o. the case if $\mu_0 = -\frac{\sigma_0^2}{2}$, $\mu_1 = -\frac{\sigma_1^2}{2}$ and $\mu_2 = -\frac{\sigma_2^2}{2}$