'Agegap' Formation hazard

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The hazard used for the formation and dissolution events (between persons i and j) has the following form:

$$\begin{aligned} h &= \exp\left(\alpha_0 + \alpha_1 P_i + \alpha_2 P_j + \alpha_3 |P_i - P_j|\right. \\ &+ \alpha_4 \left(\tfrac{A_i(t)+A_j(t)}{2}\right) \\ &+ \alpha_5 |A_i(t) - A_j(t) - D_{p,i} - \alpha_8 A_i(t)| \\ &+ \alpha_9 |A_i(t) - A_j(t) - D_{p,j} - \alpha_{10} A_j(t)| \\ &+ \left.\beta t_{diff}\right) \end{aligned}$$

The properties used are:

- α_k, β : Constants
- P_i, P_j : Number of partners of these persons
- A_i, A_j : Age
- $D_{p,i}$, $D_{p,j}$: Preferred age difference for persons i and j
- t_{diff}: Time since relationship became possible between these two persons

Here, both the age and t_{diff} contain a time dependency as follows:

$$A_i(t) = t - t_{B,i}$$

$$t_{diff} = t - t_r$$

where

- $t_{B,i}$: Time at which person i was born
- t_r : Time at which relationship became possible between these two persons

This hazard can be rewritten as follows, where the time dependency is now explicitly shown:

$$h = \exp\left(B + \alpha_4 t + \beta t + \alpha_5 |C - \alpha_8 t| + \alpha_9 |D - \alpha_{10} t|\right)$$

In this expression, B, C and D are no longer time dependent:

$$B = \alpha_0 + \alpha_1 P_i + \alpha_2 P_j + \alpha_3 |P_i - P_j| - \alpha_4 \left(\frac{t_{B,i} + t_{B,j}}{2}\right) - \beta t_r$$

$$C = (\alpha_8 - 1) t_{B,i} + t_{B,j} - D_{p,i}$$

$$D = (\alpha_{10} + 1) t_{B,j} - t_{B,i} - D_{p,j}$$

Because of the absolute values, there are four different situations we need to take into account:

1. $C > \alpha_8 t$ and $D > \alpha_{10} t$
$$h = \exp\left(B + \alpha_5 C + \alpha_9 D + t(\alpha_4 + \beta - \alpha_5\alpha_8 - \alpha_9\alpha_{10})\right)$$
2. $C > \alpha_8 t$ and $D < \alpha_{10} t$
$$h = \exp\left(B + \alpha_5 C - \alpha_9 D + t(\alpha_4 + \beta - \alpha_5\alpha_8 + \alpha_9\alpha_{10})\right)$$
3. $C < \alpha_8 t$ and $D > \alpha_{10} t$
$$h = \exp\left(B - \alpha_5 C + \alpha_9 D + t(\alpha_4 + \beta + \alpha_5\alpha_8 - \alpha_9\alpha_{10})\right)$$
4. $C < \alpha_8 t$ and $D < \alpha_{10} t$
$$h = \exp\left(B - \alpha_5 C - \alpha_9 D + t(\alpha_4 + \beta + \alpha_5\alpha_8 + \alpha_9\alpha_{10})\right)$$

For each of the cases the hazard is of the form

$$\exp(E + tF)$$

where in each of the cases:

1. $E = B + \alpha_5 C + \alpha_9 D \qquad F = \alpha_4 + \beta - \alpha_5\alpha_8 - \alpha_9\alpha_{10}$
2. $E = B + \alpha_5 C - \alpha_9 D \qquad F = \alpha_4 + \beta - \alpha_5\alpha_8 + \alpha_9\alpha_{10}$
3. $E = B - \alpha_5 C + \alpha_9 D \qquad F = \alpha_4 + \beta + \alpha_5\alpha_8 - \alpha_9\alpha_{10}$
4. $E = B - \alpha_5 C - \alpha_9 D \qquad F = \alpha_4 + \beta + \alpha_5\alpha_8 + \alpha_9\alpha_{10}$

The relevant integral for a hazard of this form is

$$\Delta T = \int_{t_0}^{t_0+\Delta t} \exp(E + Ft)dt$$

which has the analytical solution

$$\Delta T = \frac{\exp(E + Ft_0)}{F}\left(\exp(F\Delta t) - 1\right)$$

and inverse
$$\Delta t = \frac{1}{F}\ln\left[\frac{F\Delta t}{\exp(E + Ft_0)} + 1\right]$$

However, because of the absolute values in the integral, there are in general two times at which a sign inside an absolute value changes: one is when $C = \alpha_8 t$, the other when $D = \alpha_{10} t$. Calling the earliest of these times t_1^P and the other one t_2^P, the mapping from Δt to ΔT could require three different forms of the hazard:

$$\Delta T = \int_{t_0}^{t_0+\Delta t} h(t)dt = \int_{t_0}^{t_1^P} h_1(t)dt + \int_{t_1^P}^{t_2^P} h_2(t)dt + \int_{t_2^P}^{t_0+\Delta t} h_3(t)dt$$

Depending on the precise value of Δt, it could also be simpler than this.

Calculating ΔT from Δt is the most straightforward part, since the relative position of $t_0+\Delta t$ with respect to t_1^P and t_2^P is known. When starting from ΔT and solving for Δt this is no longer the case.

To solve for Δt, we first need to calculate

$$\Delta T_1^P = \int_{t_0}^{t_1^P} h_1(t)dt$$

If this value is larger than ΔT, we know that the value of $t_0 + \Delta t$ must be smaller than t_1^P and we simply need to solve

$$\Delta T = \int_{t_0}^{t_0+\Delta t} h_1(t)dt$$

In the value is smaller than ΔT, we know that $t_0 + \Delta t$ is larger than t_1^P and we need to calculate

$$\Delta T_2^P = \int_{t_1^P}^{t_2^P} h_2(t)dt$$

If ΔT is smaller than $\Delta T_1^P + \Delta T_2^P$, we know that the value of $t_0 + \Delta t$ is smaller than t_2^P and we need to solve

$$\Delta T - \Delta T_1^P = \int_{t_1^P}^{t_0+\Delta t} h_2(t)dt$$

In the final case, if ΔT is larger than $\Delta T_1^P + \Delta T_2^P$, we need to solve

$$\Delta T - \Delta T_1^P - \Delta T_2^P = \int_{t_2^P}^{t_0+\Delta t} h_3(t)dt$$