

Influence of susceptibility - probability distributions

Diana Hendrickx

February 27, 2017

If $Z \sim N(\mu, \sigma^2)$ then $E(e^z) = \int_{-\infty}^{+\infty} e^z \varphi(z) dz$ with $\varphi(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

$$\begin{aligned} E(e^z) &= \int_{-\infty}^{+\infty} e^z \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^z e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \end{aligned} \quad (1)$$

$$\begin{aligned} z - \frac{(z-\mu)^2}{2\sigma^2} &= \frac{1}{2} \left(\frac{-(z-\mu)^2}{\sigma^2} + 2z \right) \\ &= \frac{1}{2} \left(\frac{-(z-\mu)^2}{\sigma^2} + 2(z-\mu) - \sigma^2 \right) + \mu + \frac{\sigma^2}{2} \\ &= \frac{-1}{2} \left(\frac{z-\mu}{\sigma} - \sigma \right)^2 + \mu + \frac{\sigma^2}{2} \end{aligned}$$

$$\begin{aligned} (1) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left(\frac{z-\mu}{\sigma} - \sigma \right)^2} e^{\mu + \frac{\sigma^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left(\frac{z-\mu}{\sigma} - \sigma \right)^2} dz \end{aligned} \quad (2)$$

Put $w = \frac{z-\mu}{\sigma} - \sigma \Rightarrow dw = \frac{1}{\sigma} dz \Rightarrow dz = \sigma dw$.

$$\begin{aligned} (2) &= \frac{1}{\sqrt{2\pi}\sigma} e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{w^2}{2}} \sigma dw \\ &= \frac{1}{\sqrt{2\pi}} e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{w^2}{2}} dw \\ &= \frac{1}{\sqrt{2\pi}} e^{\mu + \frac{\sigma^2}{2}} \sqrt{2\pi} \\ &= e^{\mu + \frac{\sigma^2}{2}} \end{aligned}$$

$$Var(e^z) = E[e^{2z}] - (E[e^z])^2$$

$$\begin{aligned} E[e^{2z}] &= \int_{-\infty}^{+\infty} e^{2z} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} dz \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{2z - \frac{(z-\mu)^2}{2\sigma^2}} dz \end{aligned} \quad (3)$$

$$\begin{aligned} 2z - \frac{(z-\mu)^2}{2\sigma^2} &= \frac{1}{2} \left(\frac{(z-\mu)^2}{\sigma^2} + 4z \right) \\ &= \frac{1}{2} \left(-\frac{(z-\mu)^2}{\sigma^2} + 4(z-\mu) - 4\sigma^2 \right) + 2\mu + 2\sigma^2 \\ &= \frac{-1}{2} \left(\frac{z-\mu}{\sigma} - 2\sigma \right)^2 + 2\mu + 2\sigma^2 \end{aligned}$$

$$(3) = \frac{1}{\sqrt{2\pi}\sigma} e^{2\mu+2\sigma^2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \left(\frac{z-\mu}{\sigma} - 2\sigma \right)^2} dz \quad (4)$$

Put $w = \frac{z-\mu}{\sigma} - 2\sigma \Rightarrow dw = \frac{dz}{\sigma} \Rightarrow dz = \sigma dw$.

$$\begin{aligned} (4) &= \frac{1}{\sqrt{2\pi}\sigma} e^{2(\mu+\sigma^2)} \int_{-\infty}^{+\infty} e^{-\frac{w^2}{2}} \sigma dw \\ &= \frac{1}{\sqrt{2\pi}} e^{2(\mu+\sigma^2)} \int_{-\infty}^{+\infty} e^{-\frac{w^2}{2}} dw \\ &= \frac{1}{\sqrt{2\pi}} e^{2(\mu+\sigma^2)} \sqrt{2\pi} \\ &= e^{2(\mu+\sigma^2)} \end{aligned}$$

$$\begin{aligned} Var(e^z) &= e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2} \\ &= (e^{\sigma^2} - 1) e^{2\mu+\sigma^2} \end{aligned}$$

$$E(e^z) = 1 \Leftrightarrow \mu = -\frac{\sigma^2}{2}$$

$$b_0 \sim N(\mu_0, \sigma_0^2) \text{ and } b_1 \sim N(\mu_1, \sigma_1^2)$$

The following conditions have to be fulfilled:

$$E(e^{b_0+b_1}) = e^{\mu_0+\mu_1+\frac{(\sigma_0^2+\sigma_1^2)}{2}} = 1$$

$$E(e^{b_0+b_2}) = e^{\mu_0+\mu_2+\frac{(\sigma_0^2+\sigma_2^2)}{2}} = 1$$

This is a.o. the case if $\mu_0 = -\frac{\sigma_0^2}{2}$, $\mu_1 = -\frac{\sigma_1^2}{2}$ and $\mu_2 = -\frac{\sigma_2^2}{2}$