

# Constant hazard after $t_{max}$

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## 1 Motivation

In the mNRM, a function  $h$  is used to map internal time intervals  $\Delta T$  (typically sampled from an exponential distribution) to real-world time intervals  $\Delta t$ :

$$\Delta T = \int_{t_0}^{t_0 + \Delta t} h(s) ds$$

This function  $h$  is typically called the propensity function or the *hazard*. If a primitive function for  $h$  is known, evaluating  $\Delta T$  for a given  $\Delta t$  is straightforward. To calculate  $\Delta t$  from a specific value of  $\Delta T$ , the inverse of this primitive function would then be needed.

For some hazards, calculating  $\Delta T$  from  $\Delta t$  is possible, but calculating  $\Delta t$  for a given  $\Delta T$  is not. This is the case if the integral itself has an upper bound: because  $\Delta T$  can be any positive number, mapping it to a finite value of  $\Delta t$  cannot be done.

Simulations that we are interested in involve people, which are not expected to live indefinitely. For this reason it is possible to define a particular  $t_{max}$  after which the hazard just stays constant. If  $t_{max}$  is chosen so that the person involved would be 200 years old at that point, it will not make a difference for the simulation outcome as the person will be deceased long before this  $t_{max}$ . The major benefit however is that the integral will no longer have a fixed upper limit, and therefore a mapping from  $\Delta T$  to  $\Delta t$  should always be possible.

## 2 Using the modified hazard

So instead of performing calculations with the function  $h$  that we really want to use, we'll be working with a function  $g$  defined as follows:

$$g(t) = \begin{cases} h(t) & \text{if } t < t_{max} \\ h(t_{max}) & \text{if } t \geq t_{max} \end{cases}$$

The mapping between  $\Delta T$  and  $\Delta t$  is then done using this modified function  $g$ :

$$\Delta T = \int_{t_0}^{t_0 + \Delta t} g(s) ds$$

We always know the value of  $t_0$ , so if  $t_{max} < t_0$  we know that we're in the regime where we're basically using a constant hazard:

$$\Delta T = \int_{t_0}^{t_0+\Delta t} h(t_{max})ds = h(t_{max})\Delta t,$$

allowing a very straightforward mapping between the two time measures. If, on the other hand,  $t_{max} > t_0$ , we have to take care to use the modified hazard correctly.

## 2.1 Mapping $\Delta t$ to $\Delta T$

We already know that  $t_{max} > t_0$ , and in this direction, we also have  $t_0 + \Delta t$  as an input. If  $t_0 + \Delta t < t_{max}$ , then we simply need to calculate the same thing we would have calculated using the unmodified hazard:

$$\Delta T = \int_{t_0}^{t_0+\Delta t} g(s)ds = \int_{t_0}^{t_0+\Delta t} h(s)ds$$

On the other hand, if  $t_0 + \Delta t > t_{max}$ , we need to split the calculation in two:

$$\begin{aligned} \Delta T &= \int_{t_0}^{t_0+\Delta t} g(s)ds = \int_{t_0}^{t_{max}} h(s)ds + \int_{t_{max}}^{t_0+\Delta t} h(t_{max})ds \\ \Leftrightarrow \Delta T &= \int_{t_0}^{t_{max}} h(s)ds + h(t_{max})(t_0 + \Delta t - t_{max}) = \Delta T_{max} + h(t_{max})(t_0 + \Delta t - t_{max}) \end{aligned}$$

## 2.2 Mapping $\Delta T$ to $\Delta t$

In this case we have  $\Delta T$  as input and we know  $t_0$ , but we don't know yet if  $t_{max}$  is larger or smaller than  $t_0 + \Delta t$ . However we can calculate the value of the internal time interval that corresponds precisely to  $t_{max}$ , and this is in fact  $\Delta T_{max}$  from before:

$$\Delta T_{max} = \int_{t_0}^{t_{max}} h(s)ds$$

If this value is larger than the input value  $\Delta T$ , then we've actually not yet reached  $t_{max}$  and simply need to invert

$$\Delta T = \int_{t_0}^{t_0+\Delta t} h(s)ds.$$

On the other hand, if this value is smaller than  $\Delta T$ , the relevant equation again is

$$\Delta T = \int_{t_0}^{t_{max}} h(s)ds + h(t_{max})(t_0 + \Delta t - t_{max}) = \Delta T_{max} + h(t_{max})(t_0 + \Delta t - t_{max}),$$

which can be solved for  $\Delta t$ :

$$\Delta t = \frac{\Delta T - \Delta T_{max}}{h(t_{max})} + t_{max} - t_0$$